**Polly and Sam**

Polly and Sam are visited by a friend. The friend, having thought of two numbers between 2 and 800 inclusive, whispers their product to Polly and their sum to Sam. The following dialogue takes place:  
Polly: I don’t know the two numbers  
Sam: I know, and neither do I  
Polly: I know the two numbers  
Sam: So do I

What are the two numbers?

**Solution**  
There is a lot of “hidden” information in this problem. Or more accurately, it may look hidden, but it is actually in plain sight. Initially it looks impossible since it could be any two numbers and none of them have a clue what the numbers are. But when we look closer to this problem then we can start eliminating some combination of numbers until we finally see that there is only one possible option. We’ll take this in steps from the information we have:

1. We are looking for 2 numbers, and , that can each be in the range . In other words, we are looking for a pair   
   , where   
   There are possible combinations of
2. We know that Polly knows the product of these numbers and Sam knows the sum of these numbers. Both multiplication and addition are commutative operations. In other words:   
   This means that the pair would be the same as . With this almost half of the 638401 possible pairs are redundant so we can get rid of them.   
   For .  
   For .  
   For , and so on…  
   This is possible pairs remaining.
3. Now Polly says: “I don’t know the two numbers.”  
   This means a couple of things:  
   1. Both numbers cannot be prime numbers. If they were Polly would immediately know the two numbers. For example, if Polly is told that the product is 35, then she immediately knows that the two numbers are 5 and 7. By this information we can eliminate all the pairs where both and are prime numbers.  
      There are 139 prime numbers in the range . This means that we have pairs that are two prime numbers. We can eliminate all of those.  
      Now that we can eliminate 9730 pairs, we have 309870 remaining possibilities.
   2. The two numbers cannot be a prime number and that prime number squared. If is that prime number, then Polly would be told that the product is . If so the two numbers can only be either or . We have already eliminated since the order doesn’t matter due to the commutative properties of multiplication and addition. So, if Polly is given a product that is a prime number to the power of three, then she immediately knows that the two numbers are and . Since she doesn’t know the two numbers we can eliminate we can now also eliminate all pairs , where is a prime number.  
      This will eliminate 9 pairs. Now we have 309861 possible pairs remaining.
4. Next Sam says: “I know…”  
   If Sam knows that Polly doesn’t know the two numbers, then the sum that Sam is given can’t be any number:  
   1. The sum cannot be a number that could be created by adding two prime numbers. Otherwise, Sam couldn’t know that Polly didn’t know the two numbers. This means that the sum cannot be even. Goldbach conjecture says that every even integer greater than 2 can be written as the sum of two primes. While the conjecture is not proven it is known to be true for even integers up to , which is way more than 800. By this we can now also eliminate every pair that produces an even sum . Because if the sum is even, then Sam cannot know for sure that Polly didn’t know the two numbers.  
      We can now eliminate another 150399 pairs. This leaves us with 159462 possible remaining pairs.
   2. In addition to the eliminated pairs due to the sum being even, there are some odd sums that also can be eliminated for the same reason. 2 is also a prime number. If the sum is a number that is a prime number plus 2, then the two numbers could be that prime number and 2, which means that Polly from her product would know the two numbers. But since Sam knows from his sum that Polly doesn’t know the two numbers, then the sum cannot be a prime number plus 2. With this we can eliminate all the pairs that has a sum that is , where is a prime number.  
      This will eliminate 24831 pairs. We now have 134631 possible pairs remaining.
   3. The sum of the pair cannot be greater than 401.  
      If is the pair, then .  
      The reason for this is that 401 is a prime number and 401 is greater than . We have already excluded all the even sums due to Goldbach’s conjecture. Now we can also exclude all the odd sums above 401. If the sum was 403 then it could be the numbers 401 and 2, which is already excluded in b) above. If the sum is 405, then the two numbers could be 401 and 4. If Polly is told that the product is 1604, then she immediately knows that the two numbers are 401 and 4. The prime factors of 1604 are 2, 2, and 401. But since the two numbers have to be within 2 and 800, then can’t be 2 and 802. So, from the product 1604 Polly would immediately know that the two numbers are 4 and 401. Now, every odd sum above 401 can be created by , where . If Polly is told that the product is , then she immediately knows that the two numbers are and . If you move any factor from over to , then it would exceed , which is not possible. Since a pair with an odd sum greater than 401 can be created as , then Sam cannot be sure that Polly doesn’t know the two numbers if the sum is greater than 401. This now means that we can eliminate all the pairs which have a sum greater than 401.  
      With this step we can eliminate 121635 pairs. We have 12996 possible pairs remaining.
5. Sam continues to say: “…and neither do I”  
   That Sam doesn’t know the two numbers doesn’t actually give us any further information that we can use to eliminate more pairs. The only possibility that Sam would know the two numbers from the sum is 4, 5, 1599, or 1600, which would correspond to the pairs . But all of these pairs have already been eliminated above. No pairs were eliminated by this information.
6. Polly now says: “I know the two numbers.”  
   This means that the product that Polly has been given can only be created in one way from the remaining possible pairs. We can therefore eliminate all pairs that have the same product as some other pair.  
   With this step we can eliminate 8525 pairs. We have 4471 possible pairs remaining.
7. Sam says: “So do I”  
   This means in the same way that Sam must have been given a sum that can only be created in one way from the remaining pairs. We can eliminate all pairs that have the same sum as some other pair.  
   With this step we can eliminate 4470 pairs. We have only 1 possible pair remaining.
8. Now it is only one possible pair left. All the others are eliminated.  
   This pair of numbers is the answer.